Lesson 10
BME 444 - Control Systems

By the end of this lecture students will be able to:
- Write the DE of a second order system
- Determine the roots of a second order system
- Classify the output of a second order system as undamped, underdamped, critically damped, or overdamped
- Write the equation form of the step response of a second order system
- Calculate the damped frequency of an underdamped second-order system
- Calculate ζ, ω_n, ω_d, T_r, T_p, $\%OS$, and T_s of a system
- Determine values for components of a system required to meet design specifications

I. Transient analysis of a second-order system cont'd
   A. For reference, here is the LaPlace transformed version of a second-order control system we derived in the previous lesson

$$y(s) = \left[\frac{1}{\left(\frac{1}{\omega_n{}^2}\right) s^2 + \left(\frac{2\zeta}{\omega_n}\right) s + 1} \right] y_{ss} F(s)$$

   B. Second order systems can have four different types of outputs to a step input:
      1. Undamped
      2. Underdamped
      3. Critically damped
      4. Overdamped

   C. The way we determine what the response will be is to look at the *location* of the roots (aka poles) of the polynomial in the denominator of the transfer function when plotted on the *complex plane*
      1. Very quick refresher on complex numbers

D. How to find the roots of the polynomial in the denominator
   1. Use the roots([a b c]) function in MATLAB
   2. Use a version of the quadratic equation:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

E. **Important:** If any of the roots (poles) fall on the right side of the complex plane, the system will be unstable. Otherwise, there are four possibilities

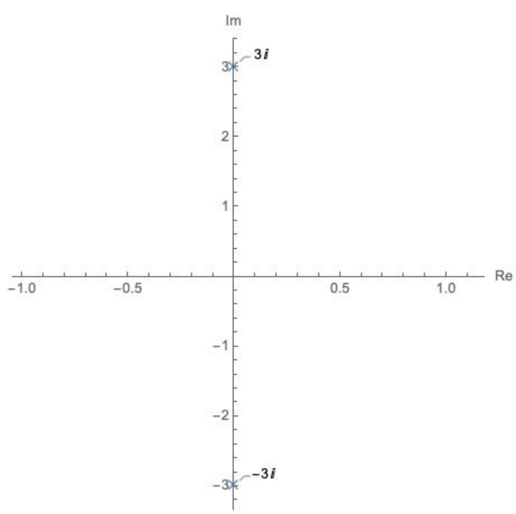
F. Undamped response ($\zeta = 0$)
   1. When $\zeta = 0$ there is no middle term in the denominator of the polynomial.
   2. Assume

$$G(s) = \frac{1}{\frac{1}{9}s^2 + 1}$$

   3. Then the roots are

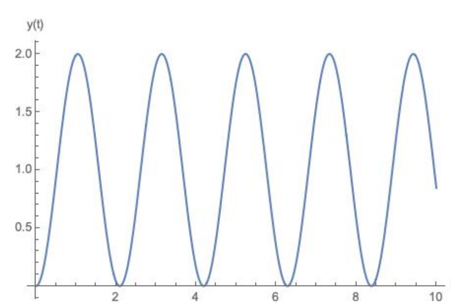
$$s_{1,2} = \pm j3$$

   4. And the plot of the poles is



   5. And the solution of the DE given a step input is:

$$y(t) = 1 - \cos(3t)$$



G. Underdamped ($0 < \zeta < 1$)

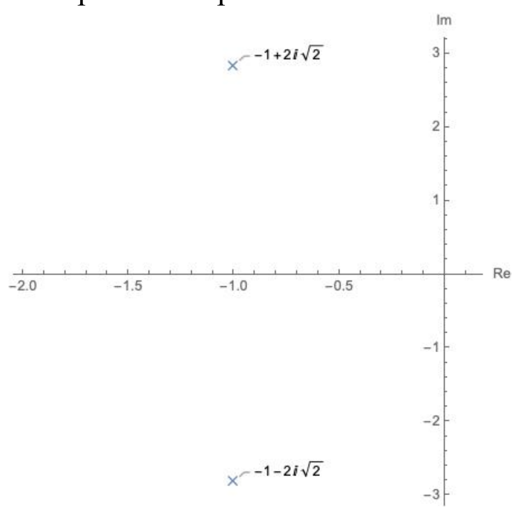
1. Assume

$$G(s) = \frac{1}{\frac{1}{9}s^2 + \frac{2}{9}s + 1}$$

2. Then the roots of the denominator are

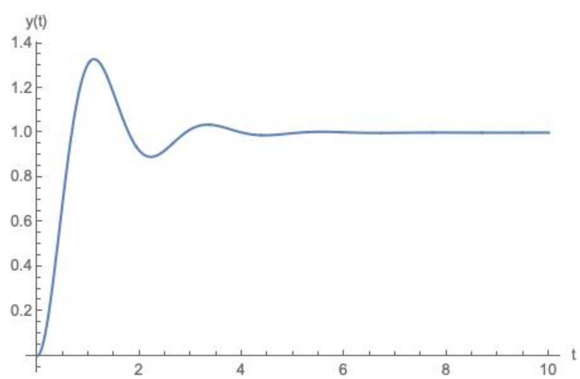
$$s_{1,2} = -1 \pm j2\sqrt{2}$$

3. The plot of the poles looks like



4. And the solution of the DE given a step input is:

$$y(t) = 1 - e^{-t}\left(\cos\left(2\sqrt{2}t\right) + \frac{\sqrt{2}}{4}\sin\left(2\sqrt{2}t\right)\right)$$



H. Critically damped ($\zeta = 1$)

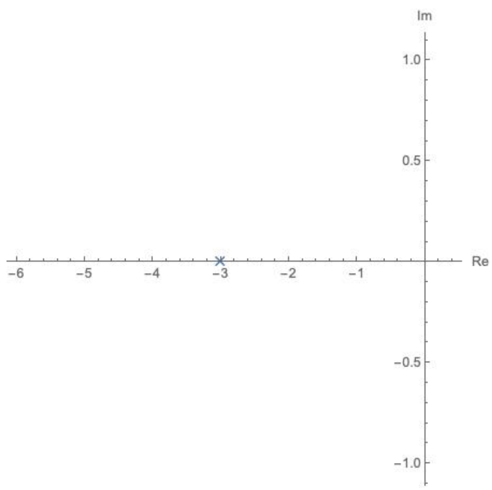
1. Assume

$$G(s) = \frac{1}{\frac{1}{9}s^2 + \frac{6}{9}s + 1}$$

2. Then the roots of the denominator are

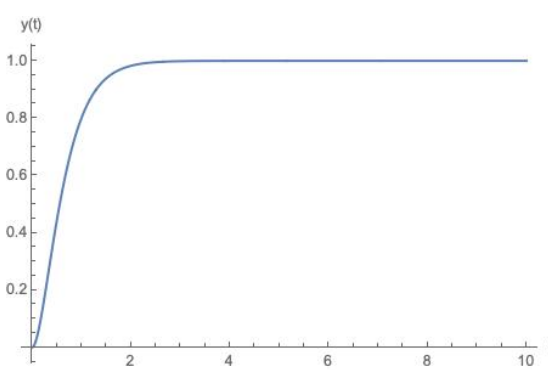
$$s_{1,2} = -3, -3$$

3. The plot of the poles looks like



4. And the solution of the DE given a step input is:

$$y(t) = 1 - 3te^{-3t} - e^{-3t}$$



I. Overdamped $(\zeta > 1)$

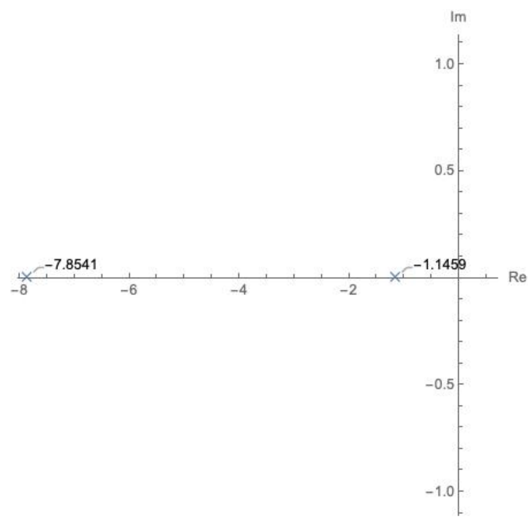
1. Assume

$$G(s) = \frac{1}{\frac{1}{9}s^2 + s + 1}$$

2. Then the roots of the denominator are

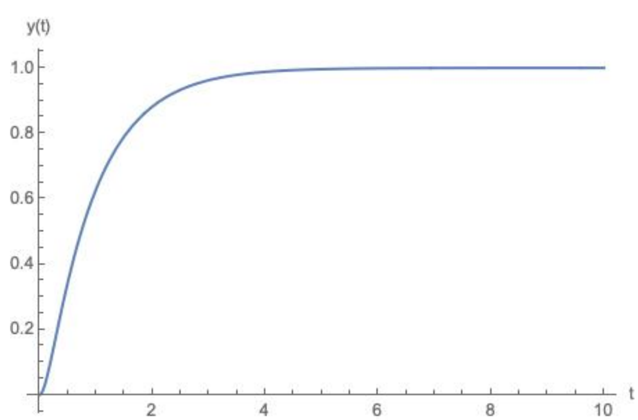
$$s_{1,2} = -7.8541, -1.1459$$

3. The plot of the poles looks like

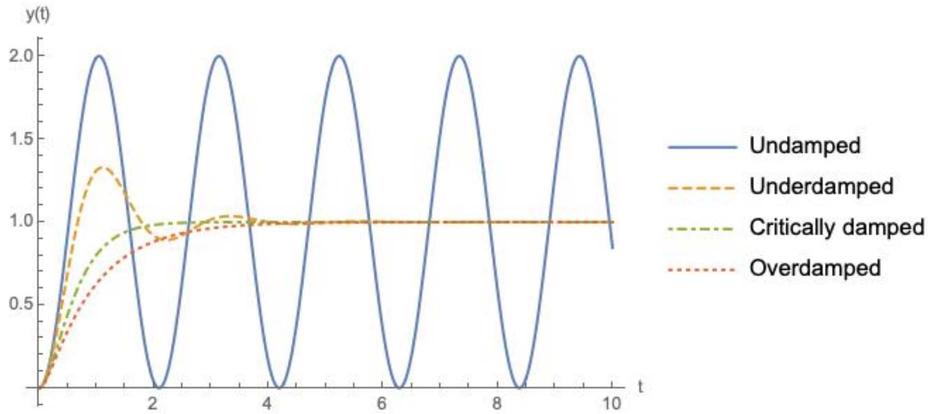


4. And the solution of the DE given a step input is:

$$y(t) = 1 + 0.171e^{-7.854t} - 1.171e^{-1.146t}$$

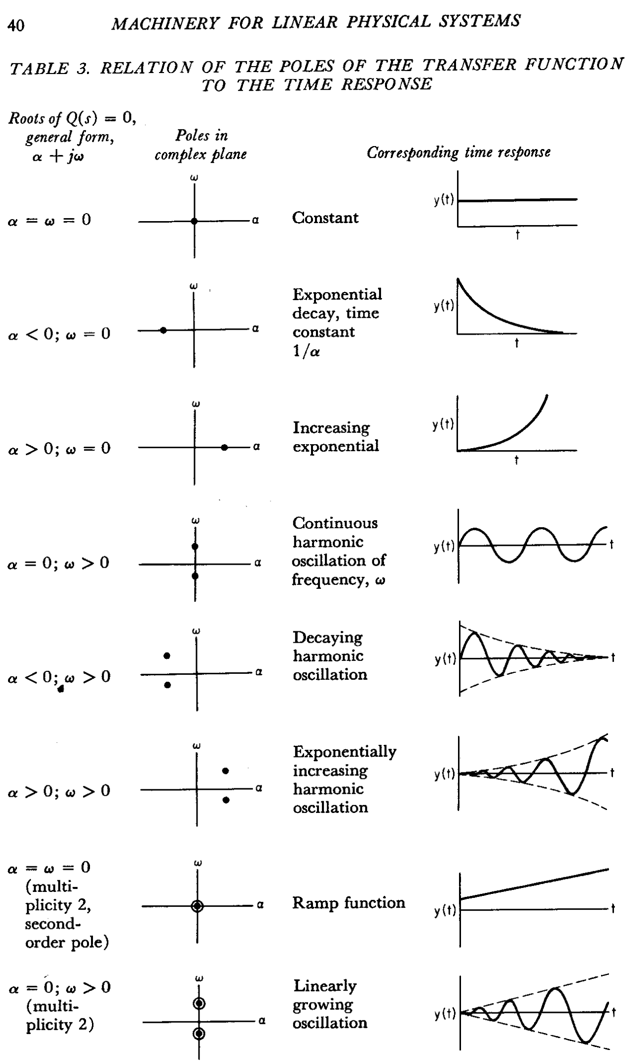


J. All four step-input responses plotted together for comparison



K. **Important.** If a system has pure imaginary roots of multiplicity one, a sinusoidal input will result in an exponentially growing sinusoidal output (instability). If a system has roots in the RHP, or imaginary roots of multiplicity greater than one, a transient response can cause instability.

L. Summary of all responses



40 *MACHINERY FOR LINEAR PHYSICAL SYSTEMS*

TABLE 3. RELATION OF THE POLES OF THE TRANSFER FUNCTION TO THE TIME RESPONSE

Roots of $Q(s) = 0$, *general form*, $\alpha + j\omega$	*Poles in complex plane*	*Corresponding time response*	
$\alpha = \omega = 0$		Constant	
$\alpha < 0;\ \omega = 0$		Exponential decay, time constant $1/\alpha$	
$\alpha > 0;\ \omega = 0$		Increasing exponential	
$\alpha = 0;\ \omega > 0$		Continuous harmonic oscillation of frequency, ω	
$\alpha < 0;\ \omega > 0$		Decaying harmonic oscillation	
$\alpha > 0;\ \omega > 0$		Exponentially increasing harmonic oscillation	
$\alpha = \omega = 0$ (multiplicity 2, second-order pole)		Ramp function	
$\alpha = 0;\ \omega > 0$ (multiplicity 2)		Linearly growing oscillation	

M. Student exercise 1.

Given: Four transfer functions

$$G(s) = \frac{400}{s^2 + 12s + 400}$$

$$G(s) = \frac{900}{s^2 + 90s + 900}$$

$$G(s) = \frac{225}{s^2 + 30s + 225}$$

$$G(s) = \frac{625}{s^2 + 625}$$

Required: Determine (1) the ζ and ω_n of each and (2) the type of response (underdamped, undamped, overdamped, critically damped)

II. Time domain solutions of the four types of responses; assumes step input; assumes 2nd order system

A. Overdamped

1. Two real poles at $-\sigma_1, -\sigma_2$
2. $c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$

B. Underdamped

1. Two complex poles at $-\sigma_d \pm j\omega_d$
2. $c(t) = Ae^{-\sigma_d t} \cos(\omega_d t - \phi)$

C. Undamped

1. Two imaginary poles at $\pm j\omega_1$
2. $c(t) = A \cos(\omega_1 t - \phi)$

D. Critically damped

1. Two real poles at $-\sigma_1$
2. $c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$

E. **Student exercise 2.** Write out the expected form of the solution to each of the transfer functions in exercise 1.

III. Characteristics of the second order transfer function

A. Natural frequency

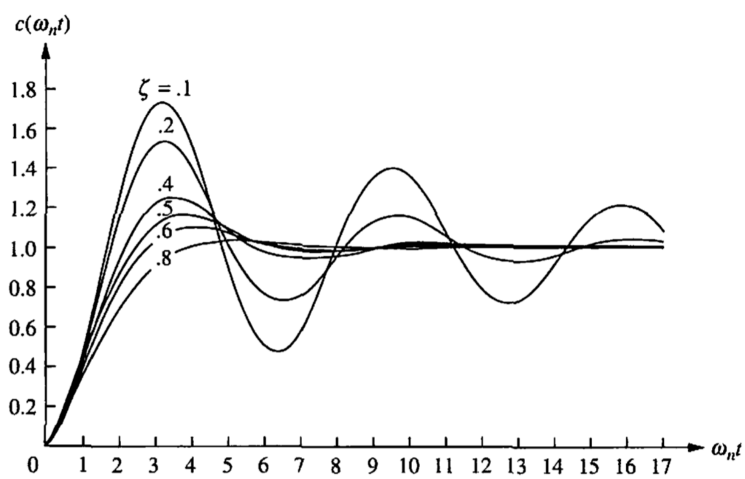
1. Represented by ω_n
2. The frequency of the transient output if it is (or if it were) undamped

B. Damped frequency

1. Represented by ω_d
2. The frequency of the output in an underdamped system
3. $\omega_d = \omega_n\sqrt{1-\zeta^2}$

C. Damping ratio

1. Represented by ζ
2. How fast an underdamped system dies down to its steady-state
3. $\zeta = \dfrac{\text{exponential decay frequency}}{\text{natural frequency}}$
4. Effect of different ζ on response



IV. Characteristics of second order systems

A. Five important features: (1) rise time, T_r, (2) peak time, T_p, (3) percent overshoot, *%OS*, (4) settling time, T_s.

B. Rise time

1. Time required for the waveform to go from $0.1y_{ss}$ (10% of steady-state output) to $0.9y_{ss}$ (90% of steady-state output).
2. No analytical solution is available. Set $y(t) = 0.9$ and $y(t)=0.1$ and solve for t.

C. Peak time

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

D. Percent overshoot

$$\%\,OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100$$

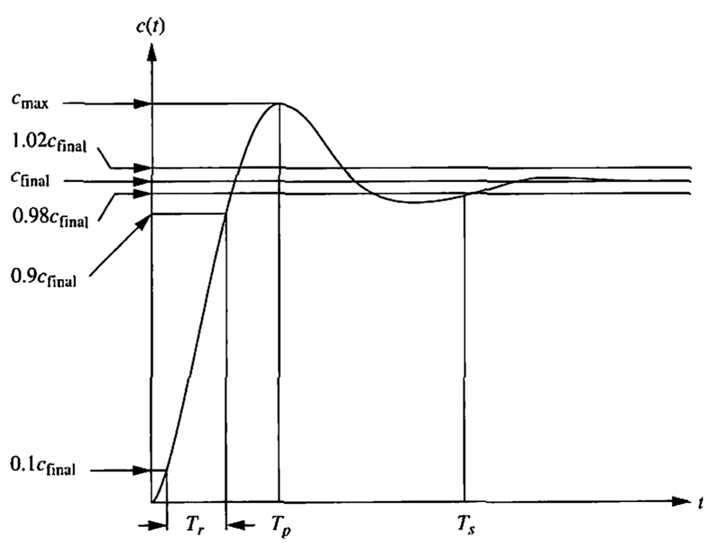
Note that you can calculate ζ if you know %OS

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

E. Settling time

$$T_s = \frac{4}{\zeta\omega_n}$$

F. Figure showing each of the five characteristics



G. **Important:** Second-order systems do not have an easily identifiable time constant like first-order systems.

H. Student exercise 3.

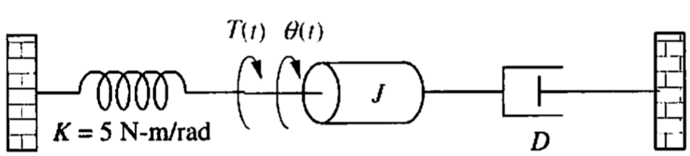
Given: The transfer function

$$G(s) = \frac{100}{s^2 + 15s + 100}$$

Required: T_p, %OS, T_s, and T_r.

I. Student exercise 4.

Given: The system shown below.



Required: Find J and D to yield 20% OS and a settling time of 2 seconds for a step input to torque.