Lesson 9
BME 444 - Control Systems

By the end of this lecture students will be able to:

- Write the DE for a first order system
- Determine the time constant, rise time, and settling time of a first-order system from its differential equation, sketch, or LaPlace transform, or vice versa
- Sketch the response of a first-order system from its equation or LaPlace transform
- Calculate the steady state value of a first-order system, given its DE or LaPlace transform
- Sketch the response of a first-order system to an impulse or step input

I. Two methods for analyzing control systems
   A. Time-domain (transient analysis)
      1. Shows how the system responds over time
   B. Frequency domain analysis
      1. Shows the steady-state response only

II. Order of the system
   A. The order of the system is the order of the differential equation that describes the system
   B. If multiple DEs are used to describe the system, then the order of the system is the sum of the highest orders from each equation
   C. We will cover the transient analysis of two orders in detail:
      1. First-order
      2. Second-order

III. Transient analysis of a first-order system DE
   A. The differential equation of a first-order system that is given an input, $F(t)$, and constant coefficients a, b, and c has the pattern

$$a\frac{dy}{dt} + by = cF(t)$$

   B. We re-write the DE above to make it easier to tell how the system will behave
   C. First change, we divide through by b and get

$$\frac{a}{b}\frac{dy}{dt} + y = \frac{c}{b}F(t)$$

D. Second, we replace *a/b* with τ and *c/b* with y_{ss}, the steady-state output of the system.

$$\tau\frac{dy}{dt} + y = y_{ss}F(t)$$

E. LaPlace transform the equation to get

$$\tau\left(sy(s) - y_0\right) + y(s) = y_{ss}F(s)$$

and rearrange to get

$$y(s) = \left(\frac{1}{\tau s + 1}\right)\left(y_{ss}F(s) + \tau y_0\right)$$

F. The solution to this differential equation will depend on the input it's given. We look at two inputs: impulse and step. **In general, the "transient response" of a first order system means its response to the unit step input.**

1. Impulse input: $F(s) = 1$

$$y(s) = \left(\frac{1}{\tau s + 1}\right)\left(y_{ss} + \tau y_0\right)$$

Inverse LaPlace to get

$$y(t) = \left(\frac{y_{ss}}{\tau} + y_0\right)e^{-\frac{t}{\tau}}$$

2. Unit step input: $F(s) = u(t) = \dfrac{1}{s}$

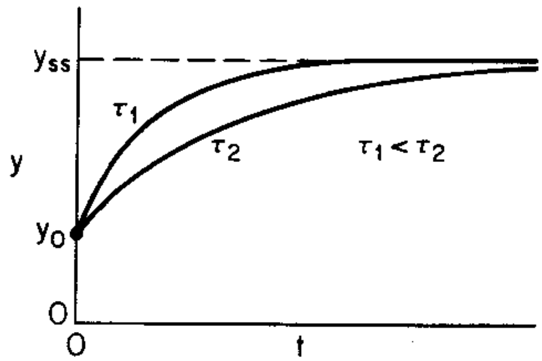
$$y(s) = \left(\frac{1}{\tau s + 1}\right)\left(\frac{y_{ss}}{s} + \tau y_0\right)$$

Inverse LaPlace to get

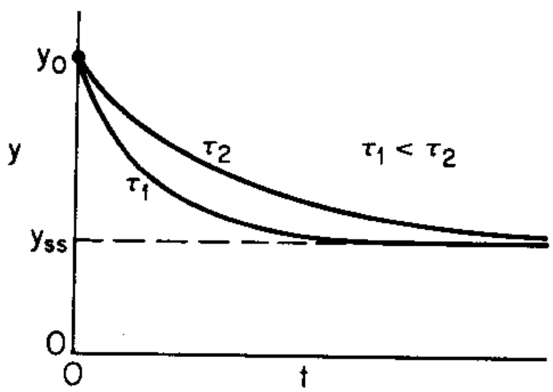
$$y(t) = y_{ss} + \left(y_0 - y_{ss}\right)e^{-\frac{t}{\tau}}$$

G. The constant y_0 is the initial condition of the system. It may be charge in a capacitor, or the initial velocity of a mass. Often, this value is 0.

1. If the system starts out with $y_0 < y_{ss}$ then the response to a unit step input looks like this:



2. If the system starts out with $y_0 > y_{ss}$ then the response to a unit step input looks like this:



H. Student example 1.

Given: A system modeled by the equation

$$3\frac{dy}{dt} + 14y = 2F(t)$$

and initial condition $y_0 = 3$

Required: Sketch the transient response to a step input

I. **Student example 2.**

Given: A system modeled by the equation

$$3\frac{dy}{dt} + 2y = 14F(t)$$

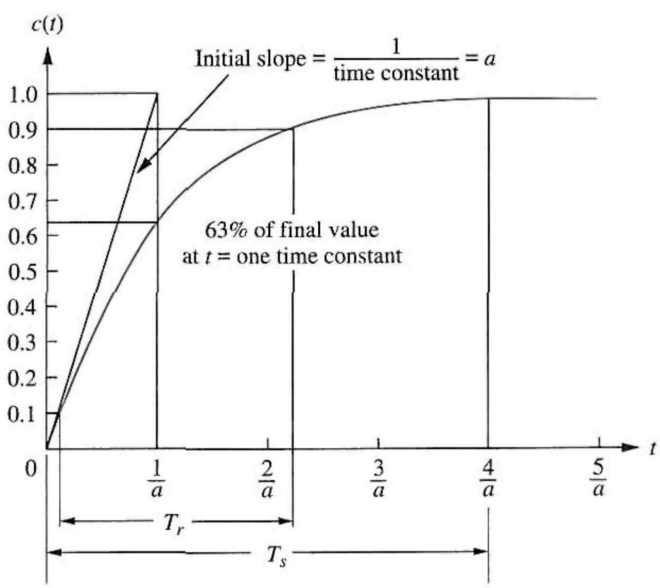
and an initial condition $y_0 = 5$

Required: Sketch the transient response to a step input

J. There are three important features of a system that you should be able to identify from the step response:

1. Time constant
2. Rise time
3. Settling time

K. Time constant



1. The time constant is a rule of thumb engineers use to predict how long it takes a system to get to steady state
2. By convention, one time constant is defined as 63% of the way to the final value
3. Here's why:

L. Rise time
   1. Time required for the system output to go from $0.1y_{ss}$ to $0.9y_{ss}$. (10% to 90% of final value)
   2. Formula

$$T_r = 2.2\tau$$

M. Settling time
   1. Time required for the system output to reach and stay within 2% of its final value.
   2. Formula

$$T_s = 4\tau$$

IV. Transient analysis of a first-order system $G(s)$

A. Often in controls we do not start with the differential equation. Instead, we start with a block diagram like this:

B. The transfer function can also be used to calculate time response and steady-state value

C. Student example 3.

Given: The transfer function below and a step input. Assume y(0) = 0.

$$G(s) = \frac{13}{s+2}$$

Required: (1) Time constant, (2) Steady-state value, (3) Rise time

D. How to get the step response of a system in MATLAB

1. Review notes from last time

E. Student example 4.

Given: The transfer function below and a step input. Assume y(0) = 0.

$$G(s) = \frac{13}{s+2}$$

Required: Plot the output response in MATLAB in two ways: (1) using tf() and (2) of the solution to the DE (inverse LaPlace).

1. Use `ilaplace(input*tf)` to get the inverse laplace transform

V. Transient analysis of a second-order system

A. The differential equation of a second-order system that is given an input, F(t), has the pattern

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = dF(t)$$

B. Constants: *a*, *b*, *c*, and *d*

C. Divide through by *c* and we get

$$\frac{a}{c}\frac{d^2y}{dt^2} + \frac{b}{c}\frac{dy}{dt} + y = \frac{d}{c}F(t)$$

D. We will now create three new substitutions:

$$\text{natural frequency} = \omega_n = \sqrt{\frac{c}{a}}$$

and

$$\text{damping ratio} = \zeta = \frac{b}{2\sqrt{ac}}$$

and

$$y_{ss} = \frac{d}{c}$$

E. Now the DE looks like this

$$\frac{1}{{\omega_n}^2}\frac{d^2y}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy}{dt} + y = y_{ss}F(t)$$

F. If we LaPlace transform, set all initial conditions = 0, and re-arrange we get

$$y(s) = \left[\frac{1}{\left(\frac{1}{\omega_n{}^2}\right) s^2 + \left(\frac{2\zeta}{\omega_n}\right) s + 1} \right] y_{ss} F(s)$$